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ELLIPTIC DIFFERENTIAL OPERATORS WITH RESPECT TO A SUBBUNDLE OF THE TANGENT BUNDLE

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In the present paper we extend the well-known theory of elliptic differential operators on compact differentiable manifolds. We define the ellipticity of differential operators with respect to a completely integrable subbundle of the tangent bundle and show that the standard argument is valid for this generalized elliptic differential operators.

1. NOTATIONS AND DEFINITIONS. Let X be a compact differentiable manifold of dimension n and let $T(X)$ be the tangent bundle of X. Let $S(X)$ be a subbundle of $T(X)$. We say that $S(X)$ is completely integrable if any local frame $\{X_1,\dots,X_m\}$ of $S(X)$ on U satisfies

$$[X_i,X_j] = \sum_{k=1}^{m} c_{ij}^{k} X_k ,$$

where c_{ij}^{k} is a C^{∞} function on U and $m = \operatorname{rank} S(X)$. If $S(X)$ is a completely integrable subbundle of $T(X)$, we can choose a coordinate system $(x_1,\dots,x_m,y_1,\dots,y_\ell)$ such that $S_p(X)$ is spanned by $\{(\partial/\partial x_1)_p,\dots,(\partial/\partial x_m)_p\}$ for all $p \in U$ (Frobenius' theorem, see [1]). Throughout this paper, we assume that a subbundle $S(X)$ of $T(X)$ is completely integrable and that a coordinate chart U is always taken as the above.

Let E be a vector bundle over X with a hermitian metric $\langle\ ,\ \rangle_E$. Let $\mathcal{E}(X,E)$ be the C^∞ sections of E over X. We write $\mathcal{E}(X) = \mathcal{E}(X,E)$ when E is the trivial line bundle. That is, $\mathcal{E}(X)$ is the space of C^∞ functions on X. We define an inner product $(\ ,\ )$ on $\mathcal{E}(X,E)$ by

$$(\xi,\eta) = \int_X \langle \xi(p),\eta(p)\rangle_E d\mu,$$

where $d\mu$ is a volume element on X. Let $W^s(E)$ be the completion of $\mathcal{E}(X,E)$ with respect to the Sobolev s-norm $\|\ \|_s$.

Let E and F be vector bundles over X with rank p and q respectively. Let

$$L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$$

be a linear mapping. We say that L is a differential operator with respect to $S(X)$ if for any choice of local coordinates and local trivializations there exists a linear partial differential operator $\tilde{L}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{E}(U)^p & \xrightarrow{\tilde{L}} & \mathcal{E}(U)^q \\
\wr\| & & \wr\| \\
\mathcal{E}(U,U\times C^p) & \longrightarrow & \mathcal{E}(U,U\times C^q) \\
\cup & & \cup \\
\mathcal{E}(X,E)|_U & \xrightarrow{L} & \mathcal{E}(X,F)|_U
\end{array}$$

commutes, where

$$\tilde{L}(f)_i = \sum_{\substack{j=1 \\ |\alpha|\leq r}}^{p} a_\alpha^{ij}(\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_m^{\alpha_m})f_j, \quad i=1,\ldots,p$$

for $f = (f_1,\ldots,f_p) \in \mathcal{E}(U)^p$. L is said to be of order r if derivatives of order $\geqq r+1$ do not appear in any local representation. We denote by $\mathrm{Diff}_r^S(E,F)$ the space of all differential operators of order r with respect to $S(X)$. We define $OP_r(E,F)$ as the space of linear mappings

$$T : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$$

such that there exists a continuous extension of T

$$T_s : W^s(E) \longrightarrow W^{s-r}(F)$$

for all $s \in \mathbb{R}$. An element of $OP_r(E,F)$ is called an operator of order r mapping E to F.

Let $L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$ be a linear mapping. We denote by L^* the adjoint of L. For any $L \in OP_r(E,F)$ there exists $L^* \in OP_r(F,E)$, and the extension

$$(L^*)_s : W^s(X,F) \longrightarrow W^{s-r}(X,E)$$

is given by the adjoint mapping

$$(L_{r-s})^* : W^s(X,F) \longrightarrow W^{s-r}(X,E).$$

Let $\pi : S'(X) \longrightarrow X$ be the bundle of nonzero cotangent vectors. π^*E and π^*F are the pull-backs of E and F over $S'(X)$ respectively. For any $r \in \mathbb{Z}$ we set

$$\mathrm{Smbl}_r^S(E,F) = \{ \sigma \in \mathrm{Hom}(\pi^*E,\pi^*F) \ ; \ \sigma(p,av) = a^r\sigma(p,v) \text{ for } (p,v) \in S'(X) \text{ and } a > 0 \}.$$

We have a linear mapping

$$\sigma_r : \mathrm{Diff}_r^S(E,F) \longrightarrow \mathrm{Smbl}_r^S(E,F).$$

We call $\sigma_r(L)$ the r-symbol of L with respect to $S(X)$.

2. PROPERTIES OF DIFFERENTIAL OPERATORS.

We summarize some properties of differential operators with respect to $S(X)$. All of them are proved in the same manner as usual differential operators (cf. [2]).

PROPOSITION 2.1. $\mathrm{Diff}_r^S(E,F) \subset OP_r(E,F)$.

PROPOSITION 2.2. *The symbol mapping* σ_r *gives an exact sequence*

$$0 \longrightarrow \mathrm{Diff}_{r-1}^S(E,F) \xrightarrow{j} \mathrm{Diff}_r^S(E,F) \xrightarrow{\sigma_r} \mathrm{Smbl}_r^S(E,F),$$

where j *is the natural inclusion.*

PROPOSITION 2.3. *Let E, F and G be vector bundles over X. If* $L_1 \in \mathrm{Diff}_k^S(E,F)$ *and* $L_2 \in \mathrm{Diff}_r^S(F,G)$, *then* $L_2 \circ L_1 \in \mathrm{Diff}_{k+r}^S(E,G)$ *and*

$$\sigma_{k+r}(L_2 \circ L_1) = \sigma_k(L_2) \cdot \sigma_r(L_1).$$

PROPOSITION 2.4. *Let* $L \in \mathrm{Diff}_r^S(E,F)$. *Then there exists* $L^* \in \mathrm{Diff}_r^S(F,E)$ *such that* $\sigma_r(L^*) = \sigma_r(L)^*$, *where* $\sigma_r(L)^*$ *is given by the adjoint mapping of the linear mapping*

$$\sigma_r(L)(p,v) : E_p \longrightarrow F_p.$$

3. PSEUDODIFFERENTIAL OPERATORS WITH RESPECT TO A SUBBUNDLE.

We take a coordinate system $(x,y) = (x_1,\ldots,x_m, y_1,\ldots,y_\ell)$ of $\mathbb{R}^n$ such that $x = (x_1,\ldots,x_m)$ and $y = (y_1,\ldots,y_\ell)$ are coordinate systems of $\mathbb{R}^m$ and $\mathbb{R}^\ell$ respectively, $m + \ell = n$.

DEFINITION 3.1. Let U be an open set in $\mathbb{R}^n$ and let r be an integer.

(a) Let $\tilde{S}_m^r(U)$ be the class of C^∞ functions $p(x,y,\xi)$ on $U \times \mathbb{R}^m$ with the following properties. For any compact set K in U and for any multiindices α, β, there exists a constant $C_{\alpha,\beta,K}$ depending on α, β, K and p such that

$$|D_{x,y}^\beta D_\xi^\alpha p(x,y,\xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{r-|\alpha|}$$

for $(x,y) \in K$ and $\xi \in \mathbb{R}^m$, where

$$D_{x,y}^\beta = (-i)^{|\beta|}\partial^{|\beta|}/\partial x_1^{\beta_1}\ldots\partial x_m^{\beta_m}\partial y_1^{\beta_{m+1}}\ldots\partial y_\ell^{\beta_{m+\ell}}$$

and

$$D_\xi^\alpha = (-i)^{|\alpha|}\partial^{|\alpha|}/\partial\xi_1^{\alpha_1}\cdots\partial\xi_m^{\alpha_m}\ .$$

(b) $S_m^r(U)$ is the set of all $p \in \tilde{S}_m^r(U)$ with the following properties. For any $\xi \neq 0$, the limit

$$\sigma_r(p)(x,y,\xi) = \lim_{\lambda\to\infty} \frac{p(x,y,\lambda\xi)}{\lambda^r}$$

exists and

$$p(x,y,\xi) - \psi(\xi)\sigma_r(p)(x,y,\xi) \in \tilde{S}_m^{r-1}(U),$$

where ψ is a C^∞ function on $\mathbb{R}^m$ with $\psi(\xi) \equiv 0$ near $\xi = 0$ and $\psi(\xi) \equiv 1$ outside the unit ball.

(c) Let $\tilde{S}_{m,0}^r(U)$ denote the set of $p \in \tilde{S}_m^r(U)$ with the following properties. There exists a compact set K in U such that for any $\xi \in \mathbb{R}^m$, the function $p_\xi(x,y) = p(x,y,\xi)$ has compact support in K. We set $S_{m,0}^r(U) = S_m^r(U) \cap \tilde{S}_{m,0}^r(U)$.

Let $\mathfrak{D}(U)$ be the space of C^∞ functions with compact support in U. For $u \in \mathfrak{D}(U)$ we set

$$\hat{u}(\xi,y) = (2\pi)^{-m}\int e^{-i\langle x,\xi\rangle} u(x,y)dx.$$

This is the Fourier transform in the x-variable. Let $p \in \tilde{S}_m^r(U)$. We define

$$L(p)u(x,y) = \int p(x,y,\xi)\hat{u}(\xi,y)e^{i\langle x,\xi\rangle}d\xi$$

for all $u \in \mathfrak{D}(U)$. Then $L(p)$ maps $\mathfrak{D}(U)$ into $\mathcal{E}(U)$ linearly. We call $L(p)$ a canonical pseudodifferential operator of order r.

<u>PROPOSITION 3.2.</u> <u>If</u> $p \in \tilde{S}_{m,0}^r(U)$, <u>then</u> $L(p)$ <u>is an</u>

<u>operator</u> <u>of</u> <u>order</u> r.

<u>PROOF.</u> Let $\hat{}^{\prime}$ be the Fourier transform in the y-variable and let $\mathfrak{F}$ be the Fourier transform in $\mathbb{R}^n$. We have

$$\mathfrak{F}(L(p)u)(\xi,\eta) = (2\pi)^{-n}\int p(x,y,\tau)\hat{u}(\tau,y)e^{-i(<x,\xi-\tau>+<y,\eta>)} d\tau dx dy.$$

Noting that

$$(2\pi)^{-\ell}\int p(x,y,\tau)\hat{u}(\tau,y)e^{-i<y,\eta>}dy$$
$$= \widehat{p\hat{u}}^{\prime}(x,\eta,\tau)$$
$$= \int \hat{p}^{\prime}(x,\eta-\omega,\tau)(\mathfrak{F}u)(\tau,\omega)d\omega,$$

we have

$$\mathfrak{F}(L(p)u)(\xi,\eta) = \int (\mathfrak{F}p)(\xi-\tau,\eta-\omega,\tau)(\mathfrak{F}u)(\tau,\omega)d\tau d\omega.$$

By (a) in DEFINITION 3.1 we obtain

$$|(\mathfrak{F}p)(\xi-\tau,\eta-\omega,\tau)| \leqq C_N(1 + |\xi - \tau|^2 + |\eta - \omega|^2)^{-N}(1 + |\tau|^2)^{\frac{r}{2}}$$

for any positive integer N. Therefore we have

$$|\mathfrak{F}(L(p)u)(\xi,\eta)| \leqq C\int (1 + |\xi - \tau|^2 + |\eta - \omega|^2)^{-N} (1 + |\tau|^2 + |\omega|^2)^{\frac{r}{2}}|\mathfrak{F}u(\tau,\omega)|d\tau d\omega.$$

If $s > 0$, then the following inequality holds

$$(1+|\xi|^2+|\eta|^2)^{\frac{s}{2}} \leqq 2^{\frac{s}{2}}(1+|\xi-\tau|^2+|\eta-\omega|^2)^{\frac{s}{2}}(1+|\tau|^2+|\omega|^2)^{\frac{s}{2}}.$$

From the above inequality we obtain

$$|\mathcal{F}(L(p)u)(\xi,\eta)|(1+|\xi|^2+|\eta|^2)^{\frac{s}{2}}$$
$$\leq C\int(1+|\xi-\tau|^2+|\eta-\omega|^2)^{-N+\frac{s}{2}}(1+|\tau|^2+|\omega|^2)^{\frac{r+s}{2}}|\mathcal{F}u(\tau,\omega)|d\tau d\omega.$$

Choosing a sufficiently large N and using Young's inequality, we obtain

$$\| L(p)u \|_s \leq C \| u \|_{s+r}.$$

The inequality for the case $s \leq 0$ is similarly obtained.

DEFINITION 3.3. Let $L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$ be a linear mapping. Then L is called a pseudodifferential operator with respect to $S(X)$ if for any coordinate chart U with trivializations of E and F over U and for any open set $U' \Subset U$ there exists a (q,p)-matrix $[p^{ij}]$, $p^{ij} \in S^r_{m,0}(U)$, so that the induced mapping

$$L_U : \mathcal{D}(U')^p \longrightarrow \mathcal{E}(U)^q$$

with $u \in \mathcal{D}(U')^p \xrightarrow{L_U} Lu$, is a (q,p)-matrix $[L(p^{ij})]$ of canonical pseudodifferential operators, where p = rank E and q = rank F.

DEFINITION 3.4. Let $L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$ be a pseudodifferential operator with respect to $S(X)$ which is represented on a coordinate chart U as in DEFINITION 3.3. The local r-symbol of L with respect to a coordinate chart U and trivializations of E and F over U is the matrix

$$\sigma_r(L)_U(x,y,\xi) = [\sigma_r(p^{ij})(x,y,\xi)].$$

Let U be an open set in $\mathbb{R}^n$. For any $p \in S^r_{m,0}(U)$ and $u \in \mathcal{D}(U)$, $L(p)u$ can be represented as

$$(1) \qquad L(p)u(x,y) = (2\pi)^{-m} \iint e^{i\langle\xi,x-z\rangle} p(x,y,\xi)u(z,y)dzd\xi.$$

Let $q(x,y;\xi;z,w)$ be a C^∞ function on $U \times \mathbb{R}^m \times U$, with compact support in the (x,y)- and (z,w)-variables, and satisfying the following two conditions.

(2.a) For any multiindices α, β and γ, there exists a constant $C_{\alpha,\beta,\gamma}$ depending on α, β, γ and q such that

$$|D_\xi^\alpha D_{x,y}^\beta D_{z,w}^\gamma q(x,y;\xi;z,w)| \leqq C_{\alpha,\beta,\gamma}(1+|\xi|)^{r-|\alpha|}.$$

(2.b) For any $\xi \neq 0$, the limit

$$\sigma_r(q)(x,y;\xi;x,y) = \lim_{\lambda\to\infty} \frac{q(x,y;\lambda\xi;x,y)}{\lambda^r}$$

exists and

$$q(x,y;\xi;x,y) - \psi(\xi)\sigma_r(q)(x,y;\xi;x,y) \in \tilde{S}_m^{r-1}(U),$$

where ψ is a C^∞ function as in DEFINITION 3.1 (b).

PROPOSITION 3.5. *Let $q(x,y;\xi;z,w)$ be a function satisfying the conditions (2.a) and (2.b), and let the operator Q be defined by*

$$(3) \qquad Qu(x,y) = (2\pi)^{-m} \iint e^{i\langle\xi,x-z\rangle} q(x,y;\xi;z,y)u(z,y)dzd\xi$$

for $u \in \mathfrak{D}(U)$. Then there exists a $p \in S_{m,0}^r(U)$ such that $Q = L(p)$ and

$$\sigma_r(p)(x,y,\xi) = \lim_{\lambda\to\infty} \frac{q(x,y;\lambda\xi;x,y)}{\lambda^r}, \quad \xi \neq 0.$$

PROOF. Let $\mathfrak{F}_z q(x,y;\xi;\zeta,w)$ be the Fourier transform of $q(x,y;\xi;z,w)$ with respect to the z-variable. Then we have

$$Qu(x,y) = \iint e^{-i<\xi,z>}\mathcal{F}_z q(x,y;\xi;\xi-\eta,y)\hat{u}(\eta,y)d\eta d\xi$$

$$= \int e^{i<\eta,x>}\{\int e^{i<\xi-\eta,x>}\mathcal{F}_z q(x,y;\xi;\xi-\eta,y)d\xi\}\hat{u}(\eta,y)d\eta.$$

If we set

$$(4) \quad p(x,y,\eta) = \int e^{i<\xi-\eta,x>}\mathcal{F}_z q(x,y;\xi;\xi-\eta,y)d\xi$$

$$= \int e^{i<\zeta,x>}\mathcal{F}_z q(x,y;\zeta+\eta;\zeta,y)d\zeta,$$

then we have a representation of Q as (1). Since q satisfies the condition (2.a), $p(x,y,\xi) \in \tilde{S}^r_{m,0}(U)$. From Taylor's theorem we obtain

$$(5) \quad \mathcal{F}_z q(x,y;\zeta+\eta;\zeta,y) = \mathcal{F}_z q(x,y;\eta;\zeta,y)$$

$$+ \sum_{|\alpha|=1} D^\alpha_\eta \mathcal{F}_z q(x,y;\eta+\zeta_0;\zeta,y)\zeta^\alpha,$$

where $\zeta_0 = \zeta_0(\zeta)$ is a point on the segment in $\mathbb{R}^m$ joining 0 to ζ. We have the following estimate by (2.a)

$$|D^\alpha_\eta \mathcal{F}_z q(x,y;\eta+\zeta_0;\zeta,y)| \leq C_N(1+|\eta+\zeta_0|)^{r-1}(1+|\zeta|)^{-N}$$

for sufficiently large N. Since $|\zeta_0| \leq |\zeta|$, we get

$$|D^\alpha_\eta \mathcal{F}_z q(x,y;\eta+\zeta_0;\zeta,y)| \leq \tilde{C}_N(1+|\eta|)^{r-1}(1+|\zeta|)^{-N+r-1}.$$

Substituting (5) in (4) and choosing N sufficiently large, we obtain

$$p(x,y,\eta) = q(x,y;\eta;x,y) + E(x,y,\eta),$$

where

$$(6) \quad |E(x,y,\eta)| \leq C(1+|\eta|)^{r-1}.$$

Therefore

$$\lim_{\lambda\to\infty} \frac{p(x,y,\lambda\eta)}{\lambda^r} = \lim_{\lambda\to\infty} \frac{q(x,y;\lambda\eta;x,y)}{\lambda^r}, \quad \eta \neq 0.$$

From the condition (2.b) and (6), it follows that

$$p(x,y,\xi) - \psi(\xi)\sigma_r(p)(x,y,\xi) \in \tilde{S}^{r-1}_m(U).$$

We rewrite (1) as

$$L(p)u(x,y) = (2\pi)^{-m} \int \{ \int e^{i<\xi,x-z>} p(x,y,\xi)d\xi \} u(z,y)dz.$$

Put

$$K(x,y,x-z) = \int e^{i<\xi,x-z>} p(x,y,\xi)d\xi.$$

Then we have the following proposition which is proved in the same manner as Proposition 3.10' in [2].

PROPOSITION 3.6. *$K(x,y,z)$ is a C^∞ function of x, y and z provided that $z \neq 0$.*

THEOREM 3.7. *Let $U \subset\subset \mathbb{R}^n$ be an open set and let $(x,x') = (x_1,\dots,x_m,x'_1,\dots,x'_\ell)$ be a coordinate system of $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^\ell$. Let $p \in S^r_{m,0}(U)$. Suppose that*

$$\tilde{F} = (F,F') : U \ni (y,y') \longmapsto (x,x') \in U$$

is a diffeomorphism of U and F' does not depend on y. For any open set $U' \subset\subset U$, we define the linear mapping

$$\tilde{L} : \mathfrak{D}(U') \longrightarrow \mathcal{E}(U)$$

by setting

$$\tilde{L}v(y,y') = L(p)(\tilde{F}^{-1})^*v(\tilde{F}(y,y')),$$

where $(\tilde{F}^{-1})^*v = v\circ\tilde{F}^{-1}$. Then there exists a function $q \in S^r_{m,0}(U)$, so that $\tilde{L} = L(q)$, and, moreover,

$$\sigma_r(q)(y,y',\xi) = \sigma_r(p)(F(y,y'),F'(y'),[{}^t(\partial F/\partial y)]^{-1}\xi).$$

PROOF. Take a function $\tilde{\psi}(x,x') \in \mathcal{D}(U)$ with $\tilde{\psi} \equiv 1$ on $\operatorname{supp} p \cup U'$. We set

$$\psi(y,y') = \tilde{\psi}(\tilde{F}(y,y')).$$

We have

$$L(p)u(x,x') = (2\pi)^{-m} \iint e^{i<\xi,x-z>} p(x,x',\xi)u(z,x')dzd\xi$$

for $u \in \mathcal{D}(U')$. We write $v(w,w') = u(\tilde{F}(w,w'))$ and $(z,z') = \tilde{F}(w,w')$. Then we obtain

$$L(p)u(\tilde{F}(y,y')) = (2\pi)^{-m} \iint e^{i<\xi,F(y,y')-F(w,y')>}$$

$$p(F(y,y'),F'(y'),\xi)u(F(w,y'),F'(y'))$$

$$|\partial F/\partial w|(w,y')dwd\xi,$$

where $|\partial F/\partial w|$ is the determinant of the Jacobian matrix $\partial F/\partial w = [\partial F_i/\partial w_j]_{i,j=1,\ldots,m}$. By the mean-value theorem we get

$$\tilde{F}(y,y') - \tilde{F}(w,w') = \tilde{H}(y,y';w,w')\begin{pmatrix} y-w \\ y'-w' \end{pmatrix},$$

where $\tilde{H}(y,y';w,w')$ is a non-singular (n,n)-matrix for (w,w') sufficiently close to (y,y'). Especially,

$$\tilde{H}(w,w';w,w') = \begin{bmatrix} \frac{\partial F}{\partial w}(w,w') & \frac{\partial F}{\partial w'}(w,w') \\ 0 & \frac{\partial F'}{\partial w'}(w') \end{bmatrix}.$$

Let $\tilde{\chi}_1(y,y';w,w')$ be a C^∞ non-negative function on $U \times U$ identically 1 near the diagonal Δ in $U \times U$ and with support on a neighbourhood of Δ where $\tilde{H}(y,y';w,w')$ is invertible. Put

$$\chi_1(y,w;y') = \tilde{\chi}_1(y,y';w,y'),$$

$$\chi_2(y,w;y') = 1 - \chi_1(y,w;y').$$

If we write

$$\tilde{H}(y,y';w,y') = \begin{bmatrix} H(y,w;y') & *_1 \\ 0 & *_2 \end{bmatrix},$$

then $H(y,w;y')$ is an (m,m)-matrix and

$$H(w,w\ ;y') = (\partial F/\partial w)(w,y').$$

Setting $\zeta = {}^tH(y,w;y')\xi$, we obtain

$$L(p)u(\tilde{F}(y,y')) = (2\pi)^{-m}\{\iint e^{i<\zeta,y-w>}p(F(y,y'),F'(y'),$$
$$[{}^tH(y,w;y')]^{-1}\zeta)\,|\partial F/\partial w|\,\psi(w,y')$$
$$\frac{\chi_1(y,w;y')}{|H(y,w;y')|}\,v(w,y')dwd\zeta$$
$$+\ Eu(\tilde{F}(y,y'))\},$$

where

$$Eu(x,x') = \iint e^{i<\xi,x-z>}p(x,x',\xi)\chi_2(F(x,x'),F(z,x');F'(x'))$$
$$u(z,x')dzd\xi.$$

We set

$$q_1(y,y';\zeta;w,w') = p(F(y,y'),F'(y'),[{}^tH(y,w;y')]^{-1}\zeta)$$
$$|\partial F/\partial w|\,(w,w')\,\frac{\chi_1(y,w;y')}{|H(y,w;y')|}\psi(w,w').$$

Then q_1 has compact support in the (y,y')- and (w,w')-variables. It is easy to check that q_1 satisfies the condition (2.a). We have

$$\sigma_r(q_1)(y,y';\zeta;y,y') = \lim_{\lambda\to\infty}\frac{q_1(y,y';\lambda\zeta;y,y')}{\lambda^r}$$

$$= \lim_{\lambda\to\infty}\frac{p(F(y,y'),F'(y'),[{}^tH(y,y;y')]^{-1}\lambda\zeta)}{\lambda^r}$$

$$= \sigma_r(p)(F(y,y'),F'(y'),[{}^t(\partial F/\partial y)]^{-1}\zeta), \quad \zeta \neq 0.$$

By the growth of

$$\psi(\xi)\sigma_r(p)(x,y,\xi) - p(x,y,\xi),$$

we see that the condition (2.b) is satisfied. We write

$$\chi_2(x,z;x') = \chi_2(F(x,x'),F(z,x');F'(x')).$$

Then we have

$$\begin{aligned} Eu(x,x') &= \iint e^{i<\xi,x-z>}p(x,x',\xi)\chi_2(x,z;x')u(z,x')dzd\xi \\ &= \int\{\int e^{i<\xi,x-z>}p(x,x',\xi)\chi_2(x,z;x')d\xi\}u(z,x')dz \\ &= \int \chi_2(x,z;x')K(x,x',x-z)u(z,x')dz \\ &= \int W(x,x',z)u(z,x')dz, \end{aligned}$$

where

$$K(x,x',w) = \int e^{i<\xi,w>}p(x,x',\xi)d\xi.$$

$K(x,x',w)$ is a C^∞ function of x, x' and w for $w \neq 0$ by PROPOSITION 3.6, moreover it has compact support in the x- and x'-variables. We write in terms of the new coordinate system (y,y'),

$$\begin{aligned} Eu(\tilde{F}(y,y')) &= \int W(F(y,y'),F'(y'),F(w,y'))u(F(w,y'),F'(y')) \\ &\qquad |\partial F/\partial w|(w,y')dw \\ &= \int W_2(y,y',w)v(w,y')dw, \end{aligned}$$

where

$$W_2(y,y',w) = W(F(y,y'),F'(y'),F(w,y'))|\partial F/\partial w|(w,y')\psi(w,y').$$

Since W_2 has compact support, we have

$$\begin{aligned} Eu(\tilde{F}(y,y')) &= \int W_2(y,y',w)v(w,y')dw \\ &= \int e^{i<y,\xi>}\{\int e^{i<w-y,\xi>}W_2(y,y',w)dw\}\hat{v}(\xi,y')d\xi \\ &= \int e^{i<y,\xi>}q_2(y,y',\xi)\hat{v}(\xi,y')d\xi, \end{aligned}$$

where

$$q_2(y,y',\xi) = \int e^{i<w-y,\xi>}W_2(y,y',w)dw,$$

and $q_2 \in \tilde{S}^t_{m,0}(U)$ for all t. This implies that $\sigma_r(q_2)(y, y',\xi) \equiv 0$. Replace $q_1(y,y';\xi;w,w')$ by $q_1(y,y',\xi)$, as given by PROPOSITION 3.5, and let $q = q_1 + q_2$. Then we have

$$L(p)u(\tilde{F}(y,y')) = L(q)v(y,y').$$

The following lemma is easily shown.

LEMMA 3.8. *Let U and V be coordinate charts of X with coordinate systems $(x,x') = (x_1,\ldots,x_m,x'_1,\ldots,x'_\ell)$ and $(y,y') = (y_1,\ldots,y_m,y'_1,\ldots,y'_\ell)$ respectively, so that, $S(X)|_U$ and $S(X)|_V$ are given by $\{\partial/\partial x_1,\ldots,\partial/\partial x_m\}$ and $\{\partial/\partial y_1,\ldots,\partial/\partial y_m\}$ respectively. Suppose that $U \cap V \neq \phi$ and that*

$$\tilde{F} = (F,F') : U\cap V \ni (y,y') \longmapsto (x,x') \in U\cap V$$

is a diffeomorphism of $U\cap V$. Then F' does not depend on y.

We can define the global symbol of a pseudodifferential operator with respect to $S(X)$ by THEOREM 3.7 and Lemma 3.8.

DEFINITION 3.9. Let $L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$ be a pseudodifferential operator with respect to $S(X)$. Then L is called a pseudodifferential operator with respect to $S(X)$ of order r if, for any coordinate chart U, the corresponding matrix of canonical pseudodifferential operators $L_U = [L(p^{ij})]$ is of order r. We denote by $\mathrm{PDiff}^S_r(E,F)$ the class of all pseudodifferential operators with respect to $S(X)$ of order r.

By THEOREM 3.7 and LEMMA 3.8 there exists a canonical linear mapping

$$\sigma_r : \mathrm{PDiff}_r^S(E,F) \longrightarrow \mathrm{Smbl}_r^S(E,F),$$

which is called the symbol mapping.

Since X is compact, we obtain the following proposition by PROPOSITION 3.2.

PROPOSITION 3.10. *If* $L \in \mathrm{PDiff}_r^S(E,F)$, *then* $L \in OP_r(E,F)$.

The following two theorems are proved by the same arguments to corresponding theorems in [2].

THEOREM 3.11. *The following sequence is exact,*

$$0 \longrightarrow K_r(E,F) \xrightarrow{j} \mathrm{PDiff}_r^S(E,F) \xrightarrow{\sigma_r} \mathrm{Smbl}_r^S(E,F) \longrightarrow 0,$$

where $K_r(E,F)$ *is the kernel of* σ_r *and* j *is the natural injection. Moreover,* $K_r(E,F) \subset OP_{r-1}(E,F)$.

THEOREM 3.12. *Let* E, F *and* G *be vector bundles over* X. *Then:*

(a) *If* $Q \in \mathrm{PDiff}_r^S(E,F)$ *and* $P \in \mathrm{PDiff}_t^S(F,G)$, *then the composition* $P \circ Q \in \mathrm{PDiff}_{r+t}^S(E,G)$ *and*

$$\sigma_{r+t}(P \circ Q) = \sigma_t(P) \cdot \sigma_r(Q).$$

(b) *If* $P \in \mathrm{PDiff}_r^S(E,F)$, *then the adjoint* P^* *of* P *exists, where* $P^* \in \mathrm{PDiff}_r^S(F,E)$, *and*

$$\sigma_r(P^*) = \sigma_r(P)^*.$$

4. ELLIPTIC DIFFERENTIAL OPERATORS.

DEFINITION 4.1. Let $s \in \mathrm{Smbl}_r^S(E,F)$. Then s is said to be elliptic if and only if for any $(p,\xi) \in S'(X)$, the linear mapping

$$s(p,\xi) : E_p \longrightarrow F_p$$

is an isomorphism.

DEFINITION 4.2. Let $L \in \mathrm{PDiff}_r^S(E,F)$. Then L is said to be elliptic if and only if $\sigma_r(L)$ is elliptic.

Let $\mathrm{PDiff}^S(E,F)$ be the direct sum $\Sigma \mathrm{PDiff}_r^S(E,F)$.

DEFINITION 4.3. Let $L \in \mathrm{PDiff}^S(E,F)$. Then $\tilde{L} \in \mathrm{PDiff}^S(F,E)$ is said to be a parametrix for L if and only if

$$L \circ \tilde{L} - I_F \in OP_{-1}(F),$$

$$\tilde{L} \circ L - I_E \in OP_{-1}(E),$$

where I_F and I_E are the identity operators on F and E respectively, and $OP_{-1}(E) = OP_{-1}(E,E)$.

Let $L \in \mathrm{Diff}_r^S(E,F)$. We set

$$\mathcal{H}_L = \{ \xi \in \mathcal{E}(X,E) \ ; \ L\xi = 0 \},$$

$$\mathcal{H}_L^{\perp} = \{ \eta \in W^0(E) \ ; \ (\xi,\eta) = 0 \ \text{ for all } \ \xi \in \mathcal{H}_L \}.$$

We write $\mathrm{Diff}_r^S(E) = \mathrm{Diff}_r^S(E,E)$.

Using results in section 3 and the standard argument for the usual case (see [2]), we obtain the following propositions and theorem.

PROPOSITION 4.4. *If $L \in \mathrm{PDiff}_r^S(E,F)$ is elliptic, then there exists a parametrix for L.*

PROPOSITION 4.5. *Let* $L \in \mathrm{Diff}_r^S(E,F)$ *be elliptic. Let* $\xi \in W^s(E)$ *have the property that* $L_s\xi = \sigma \in \mathcal{E}(X,F)$, *where* L_s *is the extension of* L *to* $W^s(E)$. *Then* $\xi \in \mathcal{E}(X,E)$.

PROPOSITION 4.6. *Let* $L \in \mathrm{Diff}_r^S(E,F)$ *be elliptic. Then, for* $\tau \in \mathcal{H}_{L^*}^{\perp} \cap \mathcal{E}(X,F)$ *there exists a unique* $\xi \in \mathcal{E}(X,E)$ *such that* $L\xi = \tau$ *and such that* ξ *is orthogonal to* $\mathcal{H}_L$ *in* $W^0(E)$.

THEOREM 4.7. *Let* $L \in \mathrm{Diff}_r^S(E)$ *be self-adjoint and elliptic. Then there exist linear mappings* H_L *and* G_L

$$H_L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,E),$$

$$G_L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,E)$$

so that

(a) $H_L(\mathcal{E}(X,E)) = \mathcal{H}_L(E)$ *and* $\dim_{\mathbb{C}} \mathcal{H}_L(E) < \infty$,

(b) $L \circ G_L + H_L = G_L \circ L + H_L = I_E$,

(c) H_L *and* G_L *are operators of order* 0,

(d) $\mathcal{E}(X,E) = \mathcal{H}_L(X,E) \oplus G_L \circ L(\mathcal{E}(X,E))$
$= \mathcal{H}_L(X,E) \oplus L \circ G_L(\mathcal{E}(X,E))$

and this decomposition is orthogonal with respect to the inner product in $W^0(E)$.

5. SMOOTH SECTIONS WITH PARAMETER SPACE. We consider the product manifold $X \times \mathbb{R}^k$ of a compact differentiable manifold X and the k-dimensional Euclidean space $\mathbb{R}^k$. Let $\rho : X \times \mathbb{R}^k \longrightarrow X$ be the projection. ρ^*E is the pull-back on $X \times \mathbb{R}^k$ of a vector bundle E over X. Then, $\mathcal{E}(X \times R^k, \rho^*E)$ is identified with the space of C^∞ sections of E on X with parameter space $\mathbb{R}^k$. Let $\mathcal{E}_0(X \times \mathbb{R}^k, \rho^*E)$

be the space of C^{∞} sections of ρ^*E over $X \times \mathbf{R}^k$ with compact support. We denote by $W^s(\rho^*E) = W^s(X \times \mathbf{R}^k, \rho^*E)$ the completion of $\mathcal{E}_0(X \times \mathbf{R}^k, \rho^*E)$ with respect to the Sobolev s-norm. Let $W(\rho^*E) = \bigcap_{s>0} W^s(\rho^*E)$. A completely integrable subbundle $S(X)$ of $T(X)$ is regarded as a subbundle of the tangent bundle $T(X \times \mathbf{R}^k)$. $\mathrm{PDiff}_r^S(\rho^*E,\rho^*F)$, $\mathrm{Diff}_r^S(\rho^*E,\rho^*F)$, ellipticity and parametrix are defined similarly. Let $L \in \mathrm{Diff}_r^S(\rho^*E,\rho^*F)$. We set

$$\tilde{\mathcal{H}}_L = \{\xi \in W(\rho^*E) \; ; \; L\xi = 0\},$$

$$\tilde{\mathcal{H}}_L^{\perp} = \{\eta \in W^0(\rho^*E) \; ; \; (\xi,\eta)_{\rho^*E} = 0 \quad \text{for all} \quad \xi \in \tilde{\mathcal{H}}_L\}.$$

$OP_r(\rho^*E,\rho^*F)$ is the space of linear mappings

$$T : \mathcal{E}_0(X \times \mathbf{R}^k, \rho^*E) \longrightarrow \mathcal{E}_0(X \times \mathbf{R}^k, \rho^*F)$$

such that there exists a continuous extension of T

$$T_s : W^s(\rho^*E) \longrightarrow W^{s-r}(\rho^*F)$$

for all $s \in \mathbf{R}$. Noting that a coordinate covering $\{U_\mu\}$ of X leads to a coordinate covering $\{U_\mu \times \mathbf{R}^k\}$ of $X \times \mathbf{R}^k$, we can rewrite propositions and theorem in section 4.

PROPOSITION 5.1. *If* $L \in \mathrm{PDiff}_r^S(\rho^*E,\rho^*F)$ *is elliptic, then there exists a parametrix for* L.

PROPOSITION 5.2. *Let* $L \in \mathrm{Diff}_r^S(\rho^*E,\rho^*F)$ *be elliptic. Let* $\xi \in W^s(\rho^*E)$ *have the property that* $L_s\xi = \sigma \in W(\rho^*F)$, *where* L_s *is the extension of* L *to* $W^s(\xi^*E)$. *Then* $\xi \in W(\rho^*E)$.

PROPOSITION 5.3. *Let* $L \in \mathrm{Diff}_r^S(\rho^*E,\rho^*F)$ *be elliptic. Then, for* $\tau \in \tilde{\mathcal{H}}_{L^*}^{\perp} \cap W(\rho^*F)$ *there exists a unique* $\xi \in W(\rho^*E)$

such that $L\xi = \tau$ *and such that* ξ *is orthogonal to* $\tilde{\mathcal{H}}_L$ *in* $W^0(\rho^*E)$.

THEOREM 5.4. *Let* $L \in \mathrm{Diff}^S_r(\rho^*E)$ *be self-adjoint and elliptic. There exist linear mappings* $\tilde{H}_L$ *and* $\tilde{G}_L$

$$\tilde{H}_L : W(\rho^*E) \longrightarrow W(\rho^*E),$$
$$\tilde{G}_L : W(\rho^*E) \longrightarrow W(\rho^*E)$$

so that

(a) $\tilde{H}_L(W(\rho^*E)) = \tilde{\mathcal{H}}_L(\rho^*E)$, $\dim_{\mathbb{C}} \tilde{\mathcal{H}}_L(\rho^*E) < \infty$,

(b) $L\circ\tilde{G}_L + \tilde{H}_L = \tilde{G}_L\circ L + \tilde{H}_L = I_E$,

(c) $\tilde{H}_L$ *and* $\tilde{G}_L$ *are operators of order* 0,

(d) $W(\rho^*E) = \tilde{\mathcal{H}}_L(\rho^*E) \oplus \tilde{G}_L\circ L(W(\rho^*E))$
$\qquad = \tilde{\mathcal{H}}_L(\rho^*E) \oplus L\circ\tilde{G}_L(W(\rho^*E))$

and this decomposition is orthogonal with respect to the inner product in $W^0(\rho^*E)$.

COROLLARY 5.5. *Let* $L \in \mathrm{Diff}^S_r(\rho^*E)$ *be self-adjoint and elliptic. Then any* $\xi \in \mathcal{E}(X \times \mathbb{R}^k, \rho^*E)$ *has a representation*

$$\xi = \eta + L\tau,$$

where $\eta, \tau \in \mathcal{E}(X \times \mathbb{R}^k, \rho^*E)$ *and* $L\eta = 0$.

PROOF. Let $\{K_i\}$ be a sequence of compact sets in $\mathbb{R}^k$ such that $K_i \Subset \mathring{K}_{i+1}$ and $\bigcup_{i=1}^{\infty} K_i = \mathbb{R}^k$. We take a function $\chi_i \in \mathcal{D}(\mathring{K}_{i+1})$ such that $\chi_i \equiv 1$ on K_i, for all i. Let $\xi_i = \chi_i\xi$. Then $\xi_i \in W(\rho^*E)$. By THEOREM 5.4, there exist uniquely $\eta_i \in \tilde{\mathcal{H}}_L(\rho^*E)$ and $\tau_i \in \tilde{G}_L(W(\rho^*E)) \subset \tilde{\mathcal{H}}_L^{\perp}$ such that

$$\xi_i = \eta_i + L\tau_i.$$

By the choice of $\{\chi_i\}$ we have

$$\xi_i = \chi_i\xi_{i+1} = \chi_i\eta_{i+1} + L(\chi_i\tau_{i+1}).$$

Since $L(\chi_i\eta_{i+1}) = \chi_i L\eta_{i+1} = 0$, we have $\chi_i\eta_{i+1} \in \tilde{\mathcal{H}}_L(\rho^*E)$. Also we see that $\chi_i\tau_{i+1} \in \tilde{\mathcal{H}}_L^{\perp}$. Therefore, we obtain by the uniqueness of the representation of ξ_i that $\eta_i = \chi_i\eta_{i+1}$ and $\tau_i = \chi_i\tau_{i+1}$. Hence limits $\eta = \lim \eta_i$ and $\tau = \lim \tau_i$ exist and $\eta, \tau \in \mathcal{E}(X \times \mathbb{R}^k, \rho^*E)$. Thus we obtain the desired representation

$$\xi = \lim \xi_i = \lim (\eta_i + L\tau_i) = \eta + L\tau.$$

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